Lecture 5

12.5 - Lines and Planes

How can we describe a line in the plane?

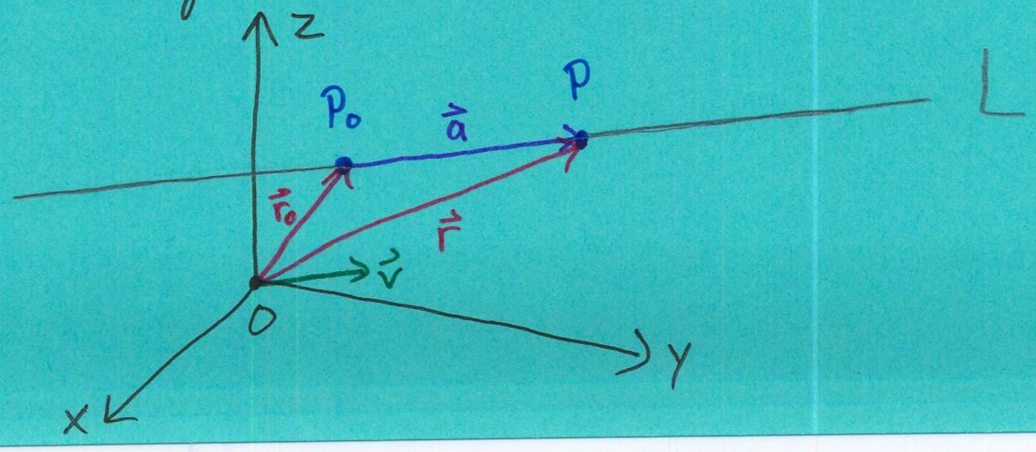
$$y - y_0 = m(x - x_0)$$

-or-

$$y = mx + b$$

Basically, a point on the line and the direction (here, the slope) of the line. We actually describe lines in $\mathbb{R}^3$ the same way: a point and a direction.

Let $P_0 = (x_0, y_0, z_0)$ be a point on the line L, and let $\vec{v}$ be a vector parallel to L. ($\vec{v} = \langle a, b, c \rangle$) Let $P = (x, y, z)$ be an arbitrary point on L. We get an equation for L as follows:



Represent points on L by their position vectors, i.e.:

P_0 as $\overrightarrow{OP_0} = \langle x_0, y_0, z_0 \rangle$ & P as $\overrightarrow{OP} = \langle x, y, z \rangle$.

Looking at the picture, we see

$$\vec{r} = \vec{r}_0 + \vec{a},$$

but $\vec{a} \parallel \vec{v}$, so $\vec{a} = t\vec{v}$ $(t \in \mathbb{R})$. Thus, the

$$\boxed{\begin{array}{l}\underline{\text{vector equation for a line}} \text{ is:} \\ \qquad \vec{r} = \vec{r}_0 + t\vec{v}\end{array}}$$

Writing this out we get

$$\langle x, y, z \rangle = \vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

which gives us the <u>parametric equations of a line</u>

$$\boxed{\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}}$$

Now, let's eliminate t from this expression:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

$$\Downarrow (a \neq 0) \qquad \Downarrow (b \neq 0) \qquad \Downarrow (c \neq 0)$$

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Since these all equal t, we have:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},}$$

called the <u>symmetric equations</u> of L.

<u>Ex 1</u> Find a vector equation and parametric equations for the line parallel to $\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$, passing through $(-7, 4, -3)$.

<u>Sol</u>: Here $\vec{r}_0 = \langle -7, 4, -3 \rangle$ & $\vec{v} = \langle 1, 4, -2 \rangle$, so

$$\vec{r} = \vec{r}_0 + t\vec{v} = \langle -7, 4, -3 \rangle + t\langle 1, 4, -2 \rangle$$
$$= \langle -7 + t, 4 + 4t, -3 - 2t \rangle$$

<u>Parametric equations</u>: $x = -7 + t$, $y = 4 + 4t$, $z = -3 - 2t$

Ex 2: Find parametric and symmetric equations for the line passing through $P=(-8,1,4)$ and $Q=(3,-2,9)$. At what point does this line intersect the xy-plane?

Sol: A vector parallel to this line is $\vec{v}=\overrightarrow{PQ}=\langle 11,-3,5\rangle$

So, the parametric ~~a vector~~ equations are

$$x=-8+11t,\ y=1-3t,\ z=4+5t$$

(note, we could have also used Q for the (x_0,y_0,z_0)).

Let's use Q for the symmetric eqn's:

$\vec{v}=\langle a,b,c\rangle=\langle 11,-3,5\rangle$ so we have

$$\frac{x-3}{11}=\frac{y+2}{-3}=\frac{z-9}{5}$$

The line hits the xy-plane when $z=0$. We can find the values of x & y in two ways: e.g.,

If $z=0$, then $z=4+5t=0 \Rightarrow t=\frac{-4}{5}$

$\Rightarrow x=-8+11t=-8+11\left(\frac{-4}{5}\right)=\frac{-84}{5}$

$\frac{y+2}{-3}=\frac{z-9}{5}=\frac{0-9}{5}=\frac{-9}{5}$

$\Rightarrow y+2=-3\left(\frac{-9}{5}\right)=\frac{27}{5}$

$\Rightarrow y=\frac{27}{5}-2=\frac{17}{5}$

So, the line hits the xy-plane at $\left(\frac{-84}{5}, \frac{17}{5}, 0\right)$. ◇

<u>Remarks</u>: i) The numbers a, b, c in the above equations are called the <u>direction numbers</u> of L

ii) If one (or two) of a, b, or c are zero, let's say $a=0$ for illustration, then the symmetric equations take the form:

$$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

(if a & $b=0$, then the equations would be $x = x_0, y = y_0, z = z$ (z can be anything))

(we cannot have $a, b, c = 0$)

This example also showed that we can write the symmetric equations as:

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$$

where (x_0, y_0, z_0) & (x_1, y_1, z_1) are points on L.

Sometimes, we just want a piece of a line, or a <u>line segment</u>. Say we want the line segment between P and Q. All we need to do is restrict the t values allowed in the vector or parametric equations: in example 2 we ~~would~~ note when

$t=0$: $x=-8+11(0)=-8$, $y=1-3(0)=1$, $z=4+5(0)=4$

$t=1$: $x=-8+11(1)=3$, $y=1-3(1)=-2$, $z=4+5(1)=9$

So letting t go from 0 to 1 gives the segment. In math terms:

vector equation: $\vec{r}(t)=\langle -8+11t,\ 1-3t,\ 4+5t\rangle,\ 0\le t \le 1$

parametric equations: $x=-8+11t,\ y=1-3t,\ z=4+5t,\ 0\le t\le 1.$

More generally, if $\vec{r}_0=\overrightarrow{OP}$ & $\vec{r}_1=\overrightarrow{OQ}$, the line is $\vec{r}=\vec{r}_0+t\vec{v}$. Note that $\vec{v}=\vec{r}_1-\vec{r}_0$, so

$$\vec{r}=\vec{r}_0+t(\vec{r}_1-\vec{r}_0)=(1-t)\vec{r}_0+t\vec{r}_1$$

So, the line segment from P to Q is given by:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1, \quad 0 \le t \le 1.$$

Ex: Show that the lines

$L_1: x = 3+2t,\ y = 4-t,\ z = 1+3t$

$L_2: x = 1+4s,\ y = 3-2s,\ z = 4+5s$

are skew, i.e., they are not parallel, and they do not intersect.

Sol: To see they are not parallel, we check if their direction vectors are parallel:

Direction vector of L_1: $\vec{v}_1 = \langle 2, -1, 3 \rangle$

" " " L_2: $\vec{v}_2 = \langle 4, -2, 5 \rangle$

We see $\vec{v}_1 \neq k\vec{v}_2$ for $k \in \mathbb{R}$, so $L_1 // L_2$.

To see if they intersect, we try to solve the system:

$$\begin{cases} 3+2t = 1+4s \\ 4-t = 3-2s \\ 1+3t = 4+5s \end{cases}$$

$$\begin{cases} 2t-4s=-2 & \Rightarrow t-2s=-1 \quad ① \\ -t+2s=-1 & \Rightarrow t-2s=1 \quad ② \\ 3t-5s=3 \quad ③ & \end{cases}$$

① & ② already contradict each other, so they don't intersect. Another way to solve this is:

② $\Rightarrow t=1+2s$, plug this into ③

$$3(1+2s)-5s=3+6s-5s=3+s=3 \Rightarrow s=0 \Rightarrow t=1$$

Since the values $t=1, s=0$ don't satisfy ①,

we see the lines dont intersect.